# Results for the structural properties of random heaps of hard disks

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The average angle of repose and the packing density of random planar heaps of hard disks falling ballistically onto a sticky base line, where the first layer of disks is quenched in random positions, are computed for heaps with a small fixed number of gaps in the base layer. The results we find appear to be almost independent of the size of the heap and they agree with those obtained from computer simulations of large systems.

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Random packing of hard spheres or disks has been studied for many years, and more recently in connection with a variety of nonequilibrium growth and aggregation models for granular media [1]. Here we study some static structural properties of a very simple kind of heap which has recently been considered by Acharyya in a numerical simulation [2]. It is a planar vertical heap made of hard disks (of unit diameter) according to the following rules. One first performs a random sequential deposition of unit segments on the line, up to saturation, and each deposited segment is chosen as the horizontal diameter of a vertical disk. These disks, frozen at their random positions, constitute the base layer of the heap. Other disks then fall downward with zero kinetic energy, and when they touch disks already settled in lower layers, seek for stable positions under gravity and then remain fixed. In the numerical simulation of Ref. [2], a typical heap contains approximately 170 disks, and statistical averages are taken on about 100 depositions. The packing density $\rho$ and the repose angle $\Theta$ have been measured, with the results $\rho=0.795\pm0.01$ and $\Theta=51.75°\pm3.00°$.

We show here that these data are in good agreement with the values we obtain for heaps of moderate size. Considering heaps where the number $N_G$ of gaps in the base layer is fixed, we perform statistical averages (involving in particular fluctuations of the total number of disks in the heap) which appear to be almost independent of $N_G$ when $N_G$ varies from 1 to 3. The repose angle appears to be practically fixed at the value $\Theta=52°$, and the density, which is more sensitive to the chosen definition for such small heaps, is found to be in the range $\rho=0.79$ and 0.82.

The computation can be illustrated on the smallest heap, shown in Figs. 1(a) and 1(b), where an arbitrary gap $R$ separates the two disks of the base layer. Case (a), where the heap contains 3 disks, corresponds to $0\leq R<R_c=\sqrt{3}-1$ and case (b), where the heap to be complete needs 6 disks, to the remaining interval $R_c\leq R<1$. (This value of $R_c$ means that heap (b) is stable for $\alpha<30°$, according to the rules given in [2]). The repose angle, which is a purely geometric factor in the absence of friction, is defined as the statistical average of $\alpha$ in case (a) and of 60° in case (b), since then the mean value of the linear fitting of the center of the disks lying on the boundary line of the heap is $\frac{1}{2}(60°+\alpha+60°-\alpha)$. Thus $\Theta=\int_0^{R_c}w(R)\alpha(R)dR+60°\int_{R_c}^{1}w(R)dR$, where $w(R)$ is the normalized probability to have a gap of size $R$ in the random sequential deposition of unit segments on the line. This car-park problem was solved [3], and, to be closer to the experimental situation, we construct the heaps with the gap distribution corresponding to the jammed configurations of the infinite line. Thus

$$w(R)=2\rho_0^{-1}\int_0^{\infty}te^{-Rt}h^2dt ,$$

where

$$h(t)=\exp\left[-\int_0^t[1-e^{-u}/u]du\right] ,$$

and

$$\rho_0=\int_0^{\infty}h^2dt\simeq0.7476$$

is the Renyi density. One then obtains $\Theta=43.894°+7.79°=51.68°$, where we indicate the contributions of cases (a) and (b), respectively. The mean value of the number of disks in the heap is 3.39.

More generally, we consider statistical heaps with a fixed number of arbitrary gaps $R_i$ on the base line; i.e., with a varying number of disks in the whole heap, since each time one gap in the distribution is greater than $R_c$, new configurations with new added disks appear, with an increasing number of layers. To each configuration cor-

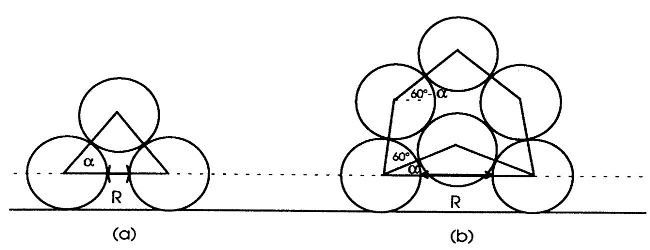


FIG. 1. The two configurations (a) and (b) of the minimal statistical heap, according to the value of the basic gap $R$ with respect to $R_c$.

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responds a mean repose angle, used to perform the statistical average with its proper weight. For example, in the case of two gaps $R_1$ and $R_2$ there are three classes *SS*, *SL*+*LS*, and *LL*, where *S* stands for short gaps ($R_i < R_c$) and *L* for large ones. These classes involve, respectively, six, 10, and 15 disks, with various equiprobable configurations in the classes where at least one *L* appears. We have carried out this geometric enumeration, before averaging, up to heaps with three gaps and 28 disks at most.

The remaining task is to obtain the probabilities for the gap distribution. One can obtain $w(R_1,R_2)$ by solving the rate equations for an appropriate hierarchic set, in the spirit of Ref. [4]. We only quote the result

$$\rho_0 w(R_1,R_2)=\int_0^\infty t^2 e^{-t(R_1+R_2)}h^2(t)dt+H(R_1,R_2)+H(R_2,R_1)$$

where

$$H(R_1,R_2)=\int_0^\infty e^{-R_1 u}h(u)\int_0^u h(v)\times e^{-R_2 v}(ve^{-v}+[(1-e^{-2v})/2])dv\,du\ ,$$

and check that $\int_0^1 dR_2 w(R_1,R_2)=w(R_1)$. We have to compute the integrals $I_1=\int_0^1 dR_2 H(R_1,R_2)$ and $I_2=\int_0^1 dR_2 H(R_2,R_1)$. $I_1$ is $\int_0^\infty e^{-uR_1}h(u)\int_0^u g(v)dv\,du$, where $g(v)=h(v)[(1-e^{-v})/2](ve^{-v}+[(1-e^{-2v})/2])$ is the derivative of $h(v)(v-[(1-e^{-2v})/2])$ since $h'(v)=[(e^{-v}-1)/v]h(v)$. The result for $I_1$ is thus $I_1=\int_0^\infty e^{-tR_1}h^2(t)(t-[(1-e^{-2t})/2])dt$. Now $I_2$ is

$$\int_0^\infty h(u)[(1-e^{-u})/u]\int_0^u h(v)\times e^{-R_1 v}(ve^{-v}+[(1-e^{-2v})/2])dv\,du\ ,$$

where a change in the order of integrations introduces $\int_v^\infty h(u)[(1-e^{-u})/u]du$ which is $h(v)$. Thus

$$\rho_0\int_0^1 dR_2 w(R_1,R_2)=I_1+I_2+\int_0^\infty e^{-tR_1}h^2(t)(t-te^{-t})dt=\rho_0 w(R_1)\ ,$$

which ensures the result. For the three gap probability we simply make the ansatz $\frac{1}{2}[w(R_1)w(R_2,R_3)+w(R_3)w(R_1,R_2)]$, as we have checked that for the observables we compute in Table I, the results are practically unchanged when $w(R_1,R_2)$ is replaced by the approximation $w(R_1)w(R_2)$.

Our results are depicted in Table I, where we detail the various components of the repose angle, which sum to $\simeq 52°$ in all cases, and the mean value *N* of the numbers of disks in the statistical heap. It can be seen from this table that the *LL*... configurations are strongly suppressed as a consequence of the shape of the probabilities $w(R_i)$ which diverge for $R_i=0$. These contact singularities, characteristic of the gap distribution at jamming [5], are here $w(R)\sim -\ln R$ and $w(R_1,R_2)\sim(R_1+R_2)^{-1}+\ln R_1 \ln R_2$. The most important contributions thus arise from the *SS*... configurations, which are the most highly ordered and not too different from regular arrangements. This observation gives some support to the estimate of the density made by Perkins [6]: he considers a regular arrangement corresponding to a constant basic gap fixed at its mean value $(1-\rho_0)/\rho_0\simeq 0.3376$. Thus $\rho=\pi/4\sin 2\Theta$ with the values $\Theta=48.02°$ and $\rho=0.7898$. When $\Theta=52°$ one finds $\rho=0.809$ for a regular heap, which is consistent with our estimate. The uncertainty we indicate for the density arises mainly from the various definitions that one may choose for a small heap where the gaps of the base layer, which are different from those in the bulk, play an important role. Inclusion of these gaps in the definition of the density gives $\simeq 0.79$, exclusion gives $\simeq 0.82$.

TABLE I. Values of the repose angle $\Theta$ and of the mean number *N* of disks for heaps with two and three gaps. The partial values of these quantities, listed in the last two columns, correspond to the configurations depicted in the second column, where the numbers of layers appear in parentheses.

| | Configuration | | Partial $\Theta$ | Partial $N$ |
|---|---|---|---|---|
| | | two gaps | | |
| | *SS* | (3) | 38.52° | 4.566 |
| $\Theta=52.00°$ | *SL*+*LS* | (4) | 12.41° | 2.196 |
| $N=7.05$ | *LL* | (5) | 1.07° | 0.291 |
| | | three gaps | | |
| | *SSS* | (4) | 33.480° | 6.622 |
| | *LSS*+*SSL* | (5) | 10.715° | 2.915 |
| | *SLS* | (5) | 5.256° | 1.433 |
| $\Theta=51.98°$ | *LSL* | (6) | 0.826° | 0.299 |
| | *SLL*+*LLS* | (6) | 1.568° | 0.599 |
| $N=11.93$ | *LLL* | (7) | 0.131° | 0.061 |

As the repose angle is defined unambiguously, we have concentrated in this work on its exact computation for some small statistical heaps. We interpret its size independence as a consequence of the particular nature of the base layer, where the random sequential deposition induces a small amount of short-range order [4].

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